Journal of Applied Mathematics and Computing
https://doi.org/10.1007/s12190-021-01523-y

ORIGINAL RESEARCH

Four classes of new entanglement-assisted quantum optimal codes

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Received: 5 January 2021 / Revised: 17 February 2021 / Accepted: 5 March 2021

Abstract

Entanglement-assisted quantum error-correcting codes (EAQECCs) provide a general framework for quantum code construction, which overcome certain self-orthogonal restriction. It becomes one main task in quantum error-correction to find EAQECCs with good parameters, especially entanglement-assisted quantum maximum distance separable (EAQMDS) codes. In this work, we construct four new families of EAQECC codes with flexible parameters in view of negacyclic codes. It is worth pointing out that those EAQECCs are EAQMDS codes when $d \leq (n+2)/2$. By exploring the selection of defining sets, the constructed EAQECCs possess larger minimum distance in contrast with the known results in the literatures.

Keywords EAQECCs · EAQMDS codes · Negacyclic codes · Defining sets

This research is supported by the National Natural Science Foundation of China (Nos. 12001002, 61772168, 61972126) and the Natural Science Foundation of Anhui Province (No. 2008085QA04).

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Published online: 13 March 2021

1 Introduction

In the past two decades, quantum error correcting codes have experienced great progress due to the conquer of decoherence and quantum noise. Based on the seminal work [1,2], Brun et al. in EAQECCs [3], a systematic mathematical method of constructing EAQECCs was proposed by using any classical linear code via an entangled-pairs share beforehand [1,2]. Besides, the bound concerning the parameters of EAQECCs has been investigated in the meanwhile [4,5].

EAQECCs contribute to making arbitrary linear codes quantized, there are abundant results in the recent years [6–8]. From the known results available in the literatures, one can see that good candidates to construct EAQECCs are constacyclic codes and generalized Reed-Solomon codes. However, an open problem is to determine the number of sharing entangled states. This problem was solved by Lu et al. who introduced the concept of the decomposition of defining sets [9]. Using a similar idea, researchers constructed many EAQMDS codes from constacyclic codes and other linear codes [10–13].

Recently, it is worth noting that a entirely new attempt on EAQMDS codes is to seek flexible parameters. In [14], Qian and Zhang obtained EAQMDS codes of length $n = q^2 + 1$ and almost EAQMDS codes of length $n = q^4 - 1$. Wang et al. derived EAQECCs of length $n = q^2 + 1$ by virtue of constacyclic codes [15]. Afterwards, abundant results of lengths $n = q^2 + 1$, $n = (q^2 + 1)/2$ and so on by constacyclic MDS codes were obtained [16,17]. These codes all have unfixed parameters and good Pauli error-correcting capacity.

Inspired by their works, we focus on constructing EAQMDS codes with variable dimension and big minimum distance. Assume that q is an odd prime power with $q \equiv \pm 2 \bmod 5$. Let $n = (q^2 + 1)/5$. Based on the analysis of the defining set, we obtain four classes of EAQECCs codes of length n as follows, especially, they are EAQMDS codes when $d \leq (n + 2)/2$:

(1) $[[n, n-2d+|D_1^1|+2, d; |D_1^1|]]_q$, where m is even integer, $q = 10\theta+3 (\theta \geq 1)$, $d = 2(mq + 3\theta + 1)$, $|D_1^1| = 4m(5m + 3) + 1$ and $1 \leq m \leq \theta$.
(2) $[[n, n-2d+|D_1^2|+2, d; |D_1^2|]]_q$, where m is odd integer, $q = 10\theta+3 (\theta \geq 1)$, $d = 2(mq + 3\theta + 1)$, $|D_1^2| = 4m(5m + 3)$ and $1 \leq m \leq \theta$.
(3) $[[n, n-2d+|D_1^3|+2, d; |D_1^3|]]_q$, where m is even integer, $q = 10\theta+7 (\theta \geq 1)$, $d = 2(mq + 7\theta + 5)$, $|D_1^3| = 4m(5m + 7) + 9$ and $1 \leq m \leq \theta$.
(4) $[[n, n-2d+|D_1^4|+2, d; |D_1^4|]]_q$, where m is odd integer, $q = 10\theta+7 (\theta \geq 1)$, $d = 2(mq + 7\theta + 5)$, $|D_1^4| = 4m(5m + 7) + 8$ and $1 \leq m \leq \theta$.

This material is organized in the following way. Section 2 gives some notations about negacyclic codes and EAQECCs. In Sect. 3, new EAQECCs and EAQMDS codes of length $n = (q^2 + 1)/5$ with variable parameters are obtained by utilizing negacyclic codes. The conclusion is summarized in Sect. 4.

2 Preliminaries

Let $\mathbb{F}_{q^2}$ denote a finite field containing q^2 elements, where q is a prime power. A k-dimensional subspace $\mathcal{C}$ of $\mathbb{F}_{q^2}^n$ having minimum Hamming distance d is called an $[n, k, d]$ linear code over $\mathbb{F}_{q^2}$. For any element $a \in \mathbb{F}_{q^2}$, denote the conjugate of a by $\overline{a} = a^q$. For any $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ and $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{F}_{q^2}^n$, their Hermitian inner product is given by

$$\langle \mathbf{u}, \mathbf{v} \rangle_H := \sum_{i=0}^{n-1} u_i \overline{v}_i = u_0 v_0^q + u_1 v_1^q + \cdots + u_{n-1} v_{n-1}^q \in \mathbb{F}_{q^2}.$$

Vectors $\mathbf{u}$ and $\mathbf{v}$ are called orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle_H = 0$. Define the Hermitian dual code of $\mathcal{C}$ as

$$\mathcal{C}^{\perp_H} := \{\mathbf{u} \in \mathbb{F}_{q^2}^n \mid \langle \mathbf{u}, \mathbf{v} \rangle_H = 0 \text{ for all } \mathbf{v} \in \mathcal{C}\}.$$

If $\mathcal{C} \subseteq \mathcal{C}^{\perp_H}$, then $\mathcal{C}$ is called Hermitian self-orthogonal. Particularly, it is called Hermitian self-dual if $\mathcal{C} = \mathcal{C}^{\perp_H}$.

Now we recall some basics of negacyclic codes. If $\mathcal{C} \subseteq \mathbb{F}_{q^2}^n$ has the property that

$$(-c_{n-1}, c_0, \ldots, c_{n-2}) \in \mathcal{C}, \text{ for all } (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C},$$

then $\mathcal{C}$ is called a negacyclic code. Usually, an element $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ is identified with its polynomial representation

$$c(x) := \sum_{i=0}^{n-1} c_i x^i = c_0 + c_1 x + \ldots + c_{n-1} x^{n-1} \in \mathbb{F}_{q^2}[x]/\langle x^n + 1 \rangle.$$

It is well known that a linear code is a negacyclic code iff the set of polynomial representation of its codewords is an ideal of the quotient ring $\mathbb{F}_{q^2}[x]/\langle x^n + 1 \rangle$. Actually, every ideal $\mathcal{C}$ of $\mathbb{F}_{q^2}[x]/\langle x^n + 1 \rangle$ is a principal ideal. Therefore, $\mathcal{C} = \langle g(x) \rangle$, where $g(x)$ is a unique monic divisor of $x^n + 1$ and has the smallest degree in $\mathcal{C}$.

Throughout this paper we let $\gcd(n, q) = 1$. Then $x^n + 1$ has no repeated root over $\mathbb{F}_{q^2}$. For any $0 \leq i \leq n - 1$. The q^2-cyclotomic coset modulo $2n$ containing i is

$$C_i := \left\{ i q^{2m} \mod 2n : 0 \leq m \leq m_i - 1 \right\},$$

where m_i is the least positive integer satisfying $i q^{2m_i} \equiv i \mod 2n$. Let ζ be a primitive $2n$-th root of unity in some extension field of $\mathbb{F}_{q^2}$. Put $\xi = \zeta^2$, then ξ is a primitive n-th root of unity. Hence,

$$x^n + 1 = \prod_{i=0}^{n-1} \left(x - \zeta \xi^i \right) = \prod_{i=0}^{n-1} \left(x - \zeta^{1+2i} \right).$$

Denote $\Omega = \{1 + 2i \mid 0 \leq i \leq n-1\}$. Let $\mathcal{C}$ be a negacyclic code of length n with generator polynomial $g(x)$. Then $D = \{i \in \Omega : g(\zeta^i) = 0\}$ is referred to as the defining set of $\mathcal{C}$ and each C_i associates with an irreducible divisor of $x^n + 1$ over $\mathbb{F}_{q^2}$. Clearly, the defining set D of $\mathcal{C}$ is a union of some q^2-cyclotomic cosets modulo $2n$. Moreover, $\dim(\mathcal{C}) = n - r$, where $r = |D|$, i.e., the cardinality of the set D. Concerning about the bound of negacyclic codes, there are the following two famous results.

Theorem 2.1 [18] *Assume that* $\gcd(n, q) = 1$. *Let* ζ *be a primitive* $2n$*-th root of unity. Let* $\mathcal{C}$ *be a negacyclic code of length* n *over* $\mathbb{F}_{q^2}$*, and its generator polynomial* $g(x)$ *have roots* ζ^{1+2i} *for* $b \leq i \leq b + \delta - 2$*, where* b *is some integer. Then the minimum distance of* $\mathcal{C}$ *is at least* δ.

Theorem 2.2 [19] *Assume* $\mathcal{C}$ *is an* $[n, k, d]$ *linear code, then*

$$n - k \geq d - 1.$$

If the equality $n - k = d - 1$ *holds, then* $\mathcal{C}$ *is called an maximum distance separable (MDS) code.*

Next, we will recall some basics about EAQECCs including the notation, bound and construction. More details please refer to [4–12,14–17].

Usually, with the help of c pairs of maximally entangled states, an $[[n, k, d; c]]_q$ entanglement-assisted quantum error-correcting code (EAQECC) can encode k information qudits into n channel qudits and correct up to $\lfloor (d-1)/2 \rfloor$ errors, where d is the minimum distance of the code. Several methods have been proposed to construct EAQECCs. And the most frequently used technique is cut apart the defining set of the codes. Based on the works before, we can employ the following mathematical scheme to construct EAQECCs.

Theorem 2.3 [20] *Let* $\mathcal{C}$ *be a negacyclic code over* $\mathbb{F}_{q^2}$ *of length* n *having defining set* D. *Suppose that* $D_1 = D \bigcap (-qD)$ *and* $D_2 = D \backslash D_1$*, where* $-qD = \{2n - qx \mid x \in D\}$. *If* $\mathcal{C}$ *has minimum Hamming distance* d*, then there exists an* $[[n, n - 2|D| + |D_1|, d; |D_1|]]_q$ *EAQECC.*

As in classical quantum codes, there is following the celebrated entanglement-assisted quantum Singleton bound.

Theorem 2.4 [2,3,5] *Assume that* $\mathcal{C}$ *is an* $[[n, k, d; c]]_q$ *entanglement-assisted quantum code. If* $d \leq (n+2)/2$*, then*

$$n + c - k \geq 2(d - 1).$$

If $\mathcal{C}$ attains the bound $n + c - k = 2(d-1)$ for $d \leq (n+2)/2$, then it is called an EAQMDS code.

3 New EAQECCs and EAQMDS codes

In this section, we are going to utilize q^2-ary negacyclic codes of length n to construct four families of new EAQECCs and EAQMDS codes with flexible parameters. For convenience to give our discussion in the sequel, it is necessary to give the following notation.

Notation 1 Let $n = (q^2+1)/5$, where $q \equiv \pm 2 \bmod 5$. According to Theorem 2.4, we always assume $d \leq (n+2)/2$ when constructing EAQMDS codes with parameters $[[n, k, d; c]]_q$. In order to ensure that q is a power of a prime, we only consider the cases of $q = 10\theta + 3$ or $q = 10\theta + 7$ ($\theta \geq 1$).

Similarly as in Lemma 4.1 from [21], we can get the following useful lemma.

Lemma 3.1 *Let $q \equiv \pm 2$ mod 5 be an odd prime power. Let $n = (q^2+1)/5$. Put $s = n/2$. Then the q^2-cyclotomic cosets modulo $2n$ from the odd integers $\{1, 3, \ldots, 2n-1\}$ are given by $C_s = \{s\}$, $C_{3s} = \{3s\}$, and $C_{s+2i} = \{s+2i, s-2i\}$, $1 \leq i \leq s-1$.*

Proof Since q has the form $q = 10\theta + 3$ or $q = 10\theta + 7$, then $4 \mid q^2 - 1$. Therefore, we have

$$sq^2 = s(q^2-1) + s = 2n \cdot \frac{q^2-1}{4} + s \equiv s \bmod 2n$$

It follows that $C_s = \{s\}$. Similarly, we can obtain $C_{3s} = \{3s\}$. For each i, $1 \leq i \leq s-1$, we have

$$(s-2i)q^2 = sq^2 - 2i(q^2+1) + 2i \equiv s + 2i \bmod 2n.$$

Besides, $(s+2i)q^2 \equiv s - 2i \bmod 2n$. This means that $C_{s+2i} = \{s+2i, s-2i\}$. We can use the same method in [21] to prove that each of the q^2-cyclotomic coset is different, where $1 \leq i \leq s-1$. Therefore, we omit it here for simplification. □

Case I $q = 10\theta + 3$

Base on the result of Lemma 3.1, we first give the following lemma which is very useful for our construction, where $C_{s+2(uq+v)}$ is an alternative expression of the q^2-cyclotomic cosets modulo $2n$ for the elements in Ω as follows:

Lemma 3.2 *Assume that $q = 10\theta + 3$ ($\theta \geq 1$) is an odd prime power. Let $n = (q^2+1)/5$. Put $s = n/2$.*

(1) When θ is even, then

$$-qC_{s+2(uq+v)} = C_{s+2(vq-u)},$$

where $1 \leq v \leq 3\theta$ and $4\theta + 2 \leq v \leq 6\theta + 1$, if $0 \leq u \leq \theta - 1$, or $1 \leq v \leq 3\theta$, if $u = \theta$.

(2) *When θ is odd, then*

$$-qC_{s+2(uq+v)} = C_{3s+2(vq-u)},$$

where $1 \le v \le 3\theta,\ 4\theta+2 \le v \le 5\theta+1$ *and* $7\theta+3 \le v \le 8\theta+2$, *if* $0 \le u \le \theta-1$, *or* $1 \le v \le 3\theta$, *if* $u = \theta$.

Proof (1) When θ is an even integer, then we have

$$C_{s+2(uq+v)} = \{s+2(uq+v), s-2(uq+v)\}$$

for $1 \le v \le 3\theta$ and $4\theta+2 \le v \le 6\theta+1$, if $0 \le u \le \theta-1$, or $1 \le v \le 3\theta$, if $u = \theta$. Since

$$\begin{aligned}-q\cdot(s-2(uq+v)) &= -qs + 2uq^2 + 2vq\\ &= -(q+1-1)\cdot s + 2u\cdot(q^2+1-1) + 2vq\\ &\equiv s + 2(vq-u) \bmod 2n.\end{aligned}$$

This gives that $-qC_{s+2(uq+v)} = C_{s+2(vq-u)}$.

(2) Similarly, when θ is an odd integer, then we have

$$C_{s+2(uq+v)} = \{s+2(uq+v), s-2(uq+v)\}$$

for $1 \le v \le 3\theta,\ 4\theta+2 \le v \le 5\theta+1$ and $7\theta+3 \le v \le 8\theta+2$, if $0 \le u \le \theta-1$, or $1 \le v \le 3\theta$, if $u = \theta$. Since

$$\begin{aligned}-q\cdot(s-2(uq+v)) &= -qs + 2uq^2 + 2vq\\ &= -(q+1-1)\cdot s + 2u\cdot(q^2+1-1) + 2vq\\ &\equiv -\frac{q-1}{2}n + 3s + 2(vq-u) \bmod 2n\\ &\equiv 3s + 2(vq-u) \bmod 2n,\end{aligned}$$

it follows that $-qC_{s+2(uq+v)} = C_{3s+2(vq-u)}$.

□

The following lemma provides a preparation for determining the number of entangled states c.

Lemma 3.3 *Assume that* $q = 10\theta + 3$ $(\theta \ge 1)$ *is an odd prime power. Let* $n = (q^2+1)/5$. *Put* $s = n/2$. *For a positive integer* $1 \le m \le \theta$, *we have*

(1) *When θ is even, let* $\alpha_i = s + 2(u_i q + v_i)$, $\beta_1 = \{m+1 \le v_1 \le 2\theta - m,\ 0 \le u_1 \le m\}$, $\beta_2 = \{m+2\theta+1 \le v_2 \le 3\theta,\ 0 \le u_2 \le m\}$, $\beta_3 = \{3\theta+1 \le v_3 \le 4\theta - m,\ 0 \le u_3 \le m-1\}$, $\beta_4 = \{m+4\theta+2 \le v_4 \le 6\theta+1-m,\ 0 \le$

$u_4 \leq m-1\}$, $\beta_5=\{m+6\theta+3 \leq v_5 \leq 8\theta+2-m,\ 0 \leq u_5 \leq m-1\}$, $\beta_6=\{m+8\theta+3 \leq v_6 \leq 10\theta+2-m, 0 \leq u_6 \leq m-1\}$.

$$T_1=\bigcup_{i=1}^{6}\bigcup_{\beta_i} C_{\alpha_i},$$

Then $T_1 \bigcap -qT_1=\emptyset$.

(2) *When* θ *is odd, let* $\alpha_i=s+2(u_iq+v_i)$, $\beta_1=\{1 \leq v_1 \leq \theta-m,\ 0 \leq u_1 \leq m\}$, $\beta_2=\{\theta+1+m \leq v_2 \leq 3\theta-m,\ 0 \leq u_2 \leq m\}$, $\beta_3=\{3\theta+2+m \leq v_3 \leq 5\theta+1-m,\ 0 \leq u_3 \leq m-1\}$, $\beta_4=\{5\theta+2+m \leq v_4 \leq 7\theta+1-m,\ 0 \leq u_4 \leq m-1\}$, $\beta_5=\{7\theta+3+m \leq v_5 \leq 9\theta+2-m,\ 0 \leq u_5 \leq m-1\}$, $\beta_6=\{9\theta+4+m \leq v_6 \leq q,\ 0 \leq u_6 \leq m-1\}$.

$$T_1=C_s\bigcup_{i=1}^{6}\bigcup_{\beta_i} C_{\alpha_i},$$

Then $T_1 \bigcap -qT_1=\emptyset$.

Proof (1) For a positive integer m with $1 \leq m \leq \theta(\theta \geq 1)$, α_i and β_i are defined as above. Let

$$\begin{aligned} T_1 &= \bigcup_{i=1}^{6}\bigcup_{\beta_i} C_{\alpha_i}, \\ &= \bigcup_{\substack{m+1\leq v_1\leq 2\theta-m,\\ 0\leq u_1\leq m}} C_{s+2(u_1q+v_1)} \bigcup_{\substack{m+2\theta+1\leq v_2\leq 3\theta,\\ 0\leq u_2\leq m}} C_{s+2(u_2q+v_2)} \\ &\quad \bigcup_{\substack{3\theta+1\leq v_3\leq 4\theta-m,\\ 0\leq u_3\leq m-1}} C_{s+2(u_3q+v_3)} \bigcup_{\substack{m+4\theta+2\leq v_4\leq 6\theta+1-m,\\ 0\leq u_4\leq m-1}} C_{s+2(u_4q+v_4)} \\ &\quad \bigcup_{\substack{m+6\theta+3\leq v_5\leq 8\theta+2-m,\\ 0\leq u_5\leq m-1}} C_{s+2(u_5q+v_5)} \bigcup_{\substack{m+8\theta+3\leq v_6\leq 10\theta+2-m,\\ 0\leq u_6\leq m-1}} C_{s+2(u_6q+v_6)}. \end{aligned}$$

Then by Lemma 3.2, we have

$$\begin{aligned} -qT_1 = & \bigcup_{\substack{m+1\leq v_1\leq 2\theta-m,\\ 0\leq u_1\leq m}} C_{s+2(v_1q-u_1))} \bigcup_{\substack{m+2\theta+1\leq v_2\leq 3\theta,\\ 0\leq u_2\leq m}} C_{s+2(v_2q+u_2)} \\ & \bigcup_{\substack{3\theta+1\leq v_3\leq 4\theta-m,\\ 0\leq u_3\leq m-1}} C_{s+2(v_3q+u_3)} \bigcup_{\substack{m+4\theta+2\leq v_4\leq 6\theta+1-m,\\ 0\leq u_4\leq m-1}} C_{s+2(v_4q+u_4)} \\ & \bigcup_{\substack{m+6\theta+3\leq v_5\leq 8\theta+2-m,\\ 0\leq u_5\leq m-1}} C_{s+2(v_5q+u_5)} \bigcup_{\substack{m+8\theta+3\leq v_6\leq 10\theta+2-m,\\ 0\leq u_6\leq m-1}} C_{s+2(v_6q+u_6)}. \end{aligned}$$

When $m+1 \le v_1 \le 2\theta - m,\ 0 \le u_1 \le m$, it follows that

$$s + 2(u_1 q + v_1) \le s + 2(mq + 2\theta - m),\ s + 2[(m+1)q - m] \le s + 2(v_1 q - u_1).$$

When $m + 2\theta + 1 \le v_2 \le 3\theta,\ 0 \le u_2 \le m$, it follows that

$$s + 2(u_2 q + v_2) \le s + 2(mq + 3\theta),\ s + 2[(2\theta + 1 + m)q - m] \le s + 2(v_2 q - u_2).$$

When $3\theta + 1 \le v_3 \le 4\theta - m,\ 0 \le u_3 \le m - 1$, it follows that

$$s + 2(u_3 q + v_3) \le s + 2[(m-1)q + 4\theta - m],\ s + 2[(3\theta + 1)q - m + 1] \le s + 2(v_3 q - u_3).$$

When $4\theta + 2 + m \le v_4 \le 6\theta + 1 - m,\ 0 \le u_4 \le m - 1$, it follows that

$$s + 2(u_4 q + v_4) \le s + 2[(m-1)q + 6\theta + 1 - m],\ s + 2[(4\theta + 2 + m)q - m + 1] \le s + 2(v_4 q - u_4).$$

When $6\theta + 3 + m \le v_5 \le 8\theta + 2 - m,\ 0 \le u_5 \le m - 1$, it follows that

$$s + 2(u_5 q + v_5) \le s + 2[(m-1)q + 8\theta + 2 - m],\ s + 2[(6\theta + 3 + m)q - m + 1] \le s + 2(v_5 q - u_5).$$

When $8\theta + 3 + m \le v_6 \le 10\theta + 2 - m,\ 0 \le u_6 \le m - 1$, it follows that

$$s + 2(u_6 q + v_6) \le s + 2[(m-1)q + 10\theta + 2 - m],\ s + 2[(8\theta + 3 + m)q - m + 1] \le s + 2(v_6 q - u_6).$$

It is easy to check that

$$\begin{aligned}
&s + 2(u_1 q + v_1) < s + 2(v_1 q - u_1), s + 2(u_1 q + v_1) < s + 2(v_2 q - u_2),\\
&s + 2(u_1 q + v_1) < s + 2(v_3 q - u_3), s + 2(u_1 q + v_1) < s + 2(v_4 q - u_4),\\
&s + 2(u_1 q + v_1) < s + 2(v_5 q - u_5), s + 2(u_1 q + v_1) < s + 2(v_6 q - u_6),\\
&s + 2(u_2 q + v_2) < s + 2(v_1 q - u_1), s + 2(u_2 q + v_2) < s + 2(v_2 q - u_2),\\
&s + 2(u_2 q + v_2) < s + 2(v_3 q - u_3), s + 2(u_2 q + v_2) < s + 2(v_4 q - u_4),\\
&s + 2(u_2 q + v_2) < s + 2(v_5 q - u_5), s + 2(u_2 q + v_2) < s + 2(v_6 q - u_6),\\
&\quad \ldots\\
&s + 2(u_6 q + v_6) < s + 2(v_1 q - u_1), s + 2(u_6 q + v_6) < s + 2(v_2 q - u_2),\\
&s + 2(u_6 q + v_6) < s + 2(v_3 q - u_3), s + 2(u_6 q + v_6) < s + 2(v_4 q - u_4),\\
&s + 2(u_6 q + v_6) < s + 2(v_5 q - u_5), s + 2(u_6 q + v_6) < s + 2(v_6 q - u_6).
\end{aligned}$$

For the range of u_i and v_i, note that $s+2(u_iq+v_i) \le (3q^2-17)/10$, $i = 1, \ldots, 6$. Moreover, the subscripts of $C_{s+2(u_iq+v_i)}$ is the least number in the set. Then,

$$T_1 \bigcap -qT_1 = \emptyset.$$

This gives the result.

(2) The proof is similar to the case (1), so we omit it here for simplification. □

Example 3.4 Let $q = 23$ and $1 \le m \le \theta = 2$. Then $n = (q^2+1)/5 = 106$, $s = n/2 = 53$ and $m = 1, 2$ respectively.

(1) Let $m = 1$. By Lemma 3.3, we can get

$$T_1 = \bigcup_{\substack{2\le v\le 3,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{v=6,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{v=7,\\ u=0}} C_{s+2(uq+v)} \bigcup_{\substack{11\le v\le 12,\\ u=0}} C_{s+2(uq+v)} \bigcup_{\substack{20\le v\le 21,\\ u=0}} C_{s+2(uq+v)}.$$

It can be verified easily that $T_1 \bigcap -qT_1 = \emptyset$.

(2) Let $m = 2$. By Lemma 3.3, we can get $T_1 = \emptyset$. It is easy to check that $T_1 \bigcap -qT_1 = \emptyset$.

Example 3.5 Let $q = 53$ and $1 \le m \le \theta = 5$. Then $n = (q^2+1)/5 = 562$, $s = n/2 = 281$ and $m = 1, 2, 3, 4, 5$ respectively.

(1) Let $m = 1$. By Lemma 3.3, we can get

$$T_1 = C_s \bigcup_{\substack{1\le v\le 4,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{7\le v\le 14,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{18\le v\le 25,\\ u=0}} C_{s+2(uq+v)} \bigcup_{\substack{28\le v\le 35,\\ u=0}} C_{s+2(uq+v)} \bigcup_{\substack{39\le v\le 46,\\ u=0}} C_{s+2(uq+v)} \bigcup_{\substack{50\le v\le 53,\\ u=0}} C_{s+2(uq+v)}.$$

It can be verified easily that $T_1 \bigcap -qT_1 = \emptyset$.

(2) Let $m = 2$. By Lemma 3.3, we can get

$$T_1 = C_s \bigcup_{\substack{1\le v\le 3,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{8\le v\le 13,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{19\le v\le 24,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{29\le v\le 34,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{40\le v\le 45,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{51\le v\le 53,\\ 0\le u\le 1}} C_{s+2(uq+v)}.$$

It can be verified easily that $T_1 \bigcap -qT_1 = \emptyset$.

(3) Let $m = 3$. By Lemma 3.3, we can get

$$T_1 = C_s \bigcup_{\substack{1 \le v \le 2,\\ 0 \le u \le 3}} C_{s+2(uq+v)} \bigcup_{\substack{9 \le v \le 12,\\ 0 \le u \le 3}} C_{s+2(uq+v)} \bigcup_{\substack{20 \le v \le 23,\\ 0 \le u \le 2}} C_{s+2(uq+v)} \bigcup_{\substack{30 \le v \le 33,\\ 0 \le u \le 2}} C_{s+2(uq+v)} \bigcup_{\substack{41 \le v \le 44,\\ 0 \le u \le 2}} C_{s+2(uq+v)} \bigcup_{\substack{52 \le v \le 53,\\ 0 \le u \le 2}} C_{s+2(uq+v)}.$$

It can be verified easily that $T_1 \bigcap -qT_1 = \emptyset$.

(4) Let $m = 4$. By Lemma 3.3, we can get

$$T_1 = C_s \bigcup_{\substack{v=1,\\ 0 \le u \le 4}} C_{s+2(uq+v)} \bigcup_{\substack{10 \le v \le 11,\\ 0 \le u \le 4}} C_{s+2(uq+v)} \bigcup_{\substack{21 \le v \le 22,\\ 0 \le u \le 3}} C_{s+2(uq+v)} \bigcup_{\substack{31 \le v \le 32,\\ 0 \le u \le 3}} C_{s+2(uq+v)} \bigcup_{\substack{42 \le v \le 43,\\ 0 \le u \le 3}} C_{s+2(uq+v)} \bigcup_{\substack{v=53,\\ 0 \le u \le 3}} C_{s+2(uq+v)}.$$

It can be verified easily that $T_1 \bigcap -qT_1 = \emptyset$.

(5) Let $m = 5$. By Lemma 3.3, we can get $T_1 = \emptyset$. It can be verified easily that $T_1 \bigcap -qT_1 = \emptyset$.

Theorem 3.6 *Assume that $q = 10\theta + 3$ $(\theta \ge 1)$ is an odd prime power. Let $n = (q^2 + 1)/5$. Put $s = n/2$. Let $\mathcal{C}$ be a negacyclic code with defining set D given as follows $D = C_s \bigcup C_{s+2} \bigcup \ldots \bigcup C_{s+2(mq+3\theta)}$, where $1 \le m \le \theta$.*

(1) When θ is even, then

$$|D_1^1| = 4m(5m + 3) + 1.$$

(2) When θ is odd, then

$$|D_1^2| = 4m(5m + 3).$$

Proof (1) Note that T_1 is defined in Lemma 3.3 and let $\alpha_i = s + 2(u_i q + v_i)$, $\beta_1' = \{0 \le v_1 \le m,\ 0 \le u_1 \le m\}$, $\beta_2' = \{2\theta + 1 - m \le v_2 \le 2\theta + m,\ 0 \le u_2 \le m\}$, $\beta_3' = \{4\theta + 1 - m \le v_3 \le 4\theta + 1 + m,\ 0 \le u_3 \le m - 1\}$, $\beta_4' = \{6\theta + 2 - m \le v_4 \le 6\theta + 2 + m,\ 0 \le u_4 \le m - 1\}$, $\beta_5' = \{8\theta + 3 - m \le v_5 \le 8\theta + 2 + m,\ 0 \le u_5 \le m - 1\}$, $\beta_6' = \{10\theta + 3 - m \le v_6 \le q,\ 0 \le u_6 \le m - 1\}$. Therefore,

$$T_1' = \bigcup_{\beta_i'} C_{\alpha_i}$$
$$= \bigcup_{\substack{0 \le v_1 \le m,\\ 0 \le u_1 \le m}} C_{s+2(u_1 q + v_1)} \bigcup_{\substack{2\theta+1-m \le v_2 \le 2\theta+m,\\ 0 \le u_2 \le m}} C_{s+2(u_2 q + v_2)}$$

$$\bigcup_{\substack{4\theta+1-m\le v_3\le 4\theta+1+m,\\ 0\le u_3\le m-1}} C_{s+2(u_3q+v_3)} \bigcup_{\substack{6\theta+2-m\le v_4\le 6\theta+2+m,\\ 0\le u_4\le m-1}} C_{s+2(u_4q+v_4)}$$
$$\bigcup_{\substack{8\theta+3-m\le v_5\le 8\theta+2+m,\\ 0\le u_5\le m-1}} C_{s+2(u_5q+v_5)} \bigcup_{\substack{10\theta+3-m\le v_6\le q,\\ 0\le u_6\le m-1}} C_{s+2(u_6q+v_6)}.$$

From Lemma 3.2, we have

$$-qT_1' = \bigcup_{\substack{0\le v_1\le m,\\ 0\le u_1\le m}} C_{s+2(v_1q-u_1)} \bigcup_{\substack{2\theta+1-m\le v_2\le 2\theta+m,\\ 0\le u_2\le m}} C_{s+2(v_2q-u_2)}$$
$$\bigcup_{\substack{4\theta+1-m\le v_3\le 4\theta+1+m,\\ 0\le u_3\le m-1}} C_{s+2(v_3q-u_3)} \bigcup_{\substack{6\theta+2-m\le v_4\le 6\theta+2+m,\\ 0\le u_4\le m-1}} C_{s+2(v_4q-u_4)}$$
$$\bigcup_{\substack{8\theta+3-m\le v_5\le 8\theta+2+m,\\ 0\le u_5\le m-1}} C_{s+2(v_5q-u_5)} \bigcup_{\substack{10\theta+3-m\le v_6\le q,\\ 0\le u_6\le m-1}} C_{s+2(v_6q-u_6)}.$$

It is easy to check that $-qT_1' = T_1'$. From the definitions of D, T_1 and T_1', we have $D = T_1 \bigcup T_1'$. Then from the definition of T_1,

$$\begin{aligned} D_1^1 &= D\bigcap(-qD) = (T_1\bigcup T_1')\bigcap(-qT_1\bigcup -qT_1') \\ &= (T_1\bigcap -qT_1)\bigcup(T_1\bigcap -qT_1')\bigcup(T_1'\bigcap -qT_1)\bigcup(T_1'\bigcap -qT_1') \\ &= T_1'. \end{aligned}$$

Therefore, $|D_1^1| = |T_1'| = 4m(5m+3)+1$.

(2) The proof is omitted since it is similar as in Case (1). □

Theorem 3.7 *Assume that $q = 10\theta+3$ ($\theta \ge 1$) is an odd prime power. Let $n = (q^2+1)/5$. Put $s = n/2$.*

- *(1) When θ is even, there are $[[n, n-2d+|D_1^1|+2, d; |D_1^1|]]_q$ EAQECCs, where $d = 2(mq+3\theta+1)$ and $1 \le m \le \theta$. Besides, they are EAQMDS codes if $d \le (n+2)/2$.*
- *(2) When θ is an odd integer, there are $[[n, n-2d+|D_1^2|+2, d; |D_1^2|]]_q$ EAQECCs, where $d = 2(mq+3\theta+1)$ and $1 \le m \le \theta$. Besides, they are EAQMDS codes if $d \le (n+2)/2$.*

Proof (1) For any fixed integer m with $1 \le m \le \theta$, assume that $\mathcal{C}$ is a negacyclic code of length $n = (q^2+1)/5$ having defining set

$$D = C_s \bigcup C_{s+2} \bigcup \dots \bigcup C_{s+2(mq+3\theta)}.$$

Then $\mathcal{C}$ has dimension $n-2(mq+3\theta)-1$. Observe that $\mathcal{C}$ have $2(mq+3\theta)+1$ consecutive zeros. By Theorem 2.1, $d(\mathcal{C}) \ge 2(mq+3\theta)+2$. From Theorem 2.2,

$\mathcal{C}$ is an $[n, n-2(mq+3\theta)-1, 2(mq+3\theta+1)]$ MDS code. From Theorem 3.6, $|D_1^1| = 4m(5m+3)+1$. By Theorem 2.3, there are $[[n, n-4[m(q-5m-3)+3\theta]-1, 2(mq+3\theta+1); 4m(5m+3)+1]]_q$ EAQECCs. It can be tested easily that

$$n+c-k = 4(mq+3\theta)+2 = 2(d-1).$$

By Theorem 2.4, the EAQECCs are MDS codes if $d \le (n+2)/2$.

(2) The proof is omitted as it is similar to the case (1). □

Example 3.8 Let $q = 23$ and $1 \le m \le \theta = 2$. Then $n = (q^2+1)/5 = 106$, $s = n/2 = 53$ and $m = 1, 2$ respectively. According to Theorem 3.7, we can obtain EAQECCs $[[106, 21, 60; 33]]_{23}$ and $[[106, 1, 106; 105]]_{23}$.

Example 3.9 Let $q = 53$ and $1 \le m \le \theta = 5$. Then $n = (q^2+1)/5 = 562$, $s = n/2 = 281$ and $m = 1, 2, 3, 4, 5$. From Theorem 3.7, we can get EAQECCs below:

$$[[562, 320, 138; 32]]_{53},\ [[562, 180, 244; 104]]_{53},\ [[562, 80, 350; 216]]_{53}.$$
$$[[562, 20, 456; 368]]_{53},\ [[562, 0, 562; 560]]_{53}.$$

Especially, they are EAQMDS codes when $d \le 282$.

Case II $q = 10\theta + 7$

In the case that the odd prime power $q = 10\theta + 7$ ($\theta \ge 1$) and $n = (q^2+1)/5$, similarly as in Case I, we can produce the following EAQECCs and EAQMDS codes. We give the following conclusions directly.

Lemma 3.10 *Assume that $q = 10\theta + 7$ ($\theta \ge 1$) is an odd prime power. Let $n = (q^2+1)/5$. Put $s = n/2$.*

(1) When θ is even, then

$$-qC_{s+2(uq+v)} = C_{s+2(vq-u)},$$

where $1 \le v \le 4\theta+2$ and $8\theta+6 \le v \le 9\theta+6$, if $0 \le u \le \theta$.

(2) When θ is odd, then

$$-qC_{s+2(uq+v)} = C_{3s+2(vq-u)},$$

where $1 \le v \le 2\theta+1$, $4\theta+2 \le v \le 5\theta+3$ and $6\theta+5 \le v \le 7\theta+4$, if $0 \le u \le \theta$.

Lemma 3.11 *Assume that $q = 10\theta + 7$ ($\theta \ge 1$) is an odd prime power. Let $n = (q^2+1)/5$. Put $s = n/2$. For any integer $1 \le m \le \theta$, we have*

(1) When θ is even, let $\alpha_i = s + 2(u_i q + v_i)$, $\beta_1 = \{m+1 \le v_1 \le 2\theta - m,\ 0 \le u_1 \le m\}$, $\beta_2 = \{m+2\theta+3 \le v_2 \le 4\theta+2-m,\ 0 \le u_2 \le m\}$, $\beta_3 = \{m+4\theta+4 \le v_3 \le 6\theta+3-m,\ 0 \le u_3 \le m\}$, $\beta_4 = \{m+6\theta+5 \le v_4 \le$

$7\theta+4,\ 0\le u_4\le m\}$, $\beta_5=\{7\theta+5\le v_5\le 8\theta+4-m,\ 0\le u_5\le m-1\}$, $\beta_6=\{m+8\theta+7\le v_6\le 10\theta+6-m, 0\le u_6\le m-1\}$.

$$T_1=\bigcup_{i=1}^{6}\bigcup_{\beta_i}C_{\alpha_i},$$

Then $T_1\bigcap -qT_1=\emptyset$.

(2) *When* θ *is odd, let* $\alpha_i=s+2(u_iq+v_i)$, $\beta_1=\{1\le v_1\le\theta-m,\ 0\le u_1\le m\}$, $\beta_2=\{\theta+2+m\le v_2\le 3\theta+1-m,\ 0\le u_2\le m\}$, $\beta_3=\{3\theta+3+m\le v_3\le 5\theta+2-m,\ 0\le u_3\le m-1\}$, $\beta_4=\{5\theta+5+m\le v_4\le 7\theta+4-m,\ 0\le u_4\le m-1\}$, $\beta_5=\{7\theta+6+m\le v_5\le 9\theta+5-m,\ 0\le u_5\le m-1\}$, $\beta_6=\{9\theta+8+m\le v_6\le q,\ 0\le u_6\le m-1\}$.

$$T_1=C_s\bigcup_{i=1}^{6}\bigcup_{\beta_i}C_{\alpha_i},$$

Then $T_1\bigcap -qT_1=\emptyset$.

Theorem 3.12 *Assume that* $q=10\theta+7$ $(\theta\ge 1)$ *is an odd prime power. Let* $n=(q^2+1)/5$. *Put* $s=n/2$. *For any integer* m *with* $1\le m\le\theta$, *let* $\mathcal{C}$ *be a negacyclic code with defining set* D *given as follows* $D=C_s\bigcup C_{s+2}\bigcup\ \dots\ \bigcup C_{s+2(mq+7\theta+4)}$.

(1) *When* θ *is even, then*

$$|D_1^3|=4m(5m+7)+9.$$

(2) *When* θ *is odd, then*

$$|D_1^4|=4m(5m+7)+8.$$

Theorem 3.13 *Assume that* $q=10\theta+7$ $(\theta\ge 1)$ *is an odd prime power. Let* $n=(q^2+1)/5$. *Put* $s=n/2$.

(1) *When* θ *is even, there are* $[[n,n-2d+|D_1^3|+2,d;|D_1^3|]]_q$ *EAQECCs, where* $d=2(mq+7\theta+5)$ *and* $1\le m\le\theta$. *Besides, they are EAQMDS codes if* $d\le(n+2)/2$.
(2) *When* θ *is odd, there are* $[[n,n-2d+|D_1^4|+2,d;|D_1^4|]]_q$ *EAQECCs, where* $d=2(mq+7\theta+5)$ *and* $1\le m\le\theta$. *Besides, they are EAQMDS codes if* $d\le(n+2)/2$.

Example 3.14 Take $q=37$ and $1\le m\le\theta=3$, then $n=(q^2+1)/5=274$, $s=n/2=137$ and $m=1,2,3$.

(1) Let $m = 1$. In the light of Lemma 3.11, we can get

$$T_1 = C_s \bigcup_{\substack{1\le v\le 2,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{6\le v\le 9,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{13\le v\le 16,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{21\le v\le 24,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{28\le v\le 31,\\ u=0}} C_{s+2(uq+v)} \bigcup_{\substack{36\le v\le 37,\\ u=0}} C_{s+2(uq+v)}.$$

It can be examined that $T_1 \bigcap -qT_1 = \emptyset$. Then from Theorem 3.13, we can get an EAQMDS code $[[274, 80, 126; 56]]_{37}$.

(2) Let $m = 2$. In the light of Lemma 3.11, we can get

$$T_1 = C_s \bigcup_{\substack{v=1,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{7\le v\le 8,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{14\le v\le 15,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{22\le v\le 23,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{29\le v\le 30,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{v=37,\\ 0\le u\le 1}} C_{s+2(uq+v)}.$$

It can be examined that $T_1 \bigcap -qT_1 = \emptyset$. Then from Theorem 3.13, we can get an EAQECC code $[[274, 20, 200; 144]]_{37}$.

(3) Let $m = 3$. In the light of Lemma 3.11, we can get $T_1 = \emptyset$. It can be examined that $T_1 \bigcap -qT_1 = \emptyset$. Then from Theorem 3.13, we can get an EAQECC code $[[274, 0, 274; 272]]_{37}$.

Example 3.15 Take $q = 47$ and $1 \le m \le k = 4$, then $n = (q^2 + 1)/5 = 442$, $s = n/2 = 221$ and $m = 1, 2, 3, 4$ respectively.

(1) Let $m = 1$. In the light of Lemma 3.11, we have

$$T_1 = \bigcup_{\substack{2\le v\le 7,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{12\le v\le 17,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{21\le v\le 26,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{30\le v\le 32,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{33\le v\le 35,\\ u=0}} C_{s+2(uq+v)} \bigcup_{\substack{40\le v\le 45,\\ u=0}} C_{s+2(uq+v)}.$$

It can be examined that $T_1 \bigcap -qT_1 = \emptyset$. By Theorem 3.13, we can derive an EAQMDS code $[[442, 181, 160; 57]]_{47}$.

(2) Let $m = 2$. In the light of Lemma 3.11, we can derive

$$T_1 = \bigcup_{\substack{3\le v\le 6,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{13\le v\le 16,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{22\le v\le 25,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{31\le v\le 32,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{33\le v\le 34,\\ 0\le u\le 1}} C_{s+2(uq+v)} \bigcup_{\substack{41\le v\le 44,\\ 0\le u\le 1}} C_{s+2(uq+v)}.$$

It can be examined that $T_1 \bigcap -qT_1 = \emptyset$. Then from Theorem 3.13, we can derive an EAQECC code $[[442, 81, 254; 145]]_{47}$.

(3) Let $m = 3$. In the light of Lemma 3.11, we can derive

$$T_1 = \bigcup_{\substack{4\le v\le 5,\\ 0\le u\le 3}} C_{s+2(uq+v)} \bigcup_{\substack{14\le v\le 15,\\ 0\le u\le 3}} C_{s+2(uq+v)} \bigcup_{\substack{23\le v\le 24,\\ 0\le u\le 3}} C_{s+2(uq+v)} \bigcup_{\substack{v=32,\\ 0\le u\le 3}} C_{s+2(uq+v)} \bigcup_{\substack{v=33,\\ 0\le u\le 2}} C_{s+2(uq+v)} \bigcup_{\substack{42\le v\le 43,\\ 0\le u\le 2}} C_{s+2(uq+v)}.$$

It can be checked that $T_1 \bigcap -qT_1 = \emptyset$. From Theorem 3.13, we can derive an EAQECC code $[[442, 21, 348; 273]]_{47}$.

(4) Let $m = 4$. In the light of Lemma 3.11, we can derive $T_1 = \emptyset$. It can be checked that $T_1 \bigcap -qT_1 = \emptyset$. From Theorem 3.13, we can derive an EAQECC code $[[442, 1, 442; 441]]_{47}$.

4 Conclusion

In this paper, four classes of EAQECCs and EAQMDS codes on length $n = (q^2+1)/5$ are built in view of negacyclic codes through cutting apart their defining sets. In contrast with the known EAQECCs with the same length, our EAQECCs have larger minimum distance and variable dimension. A further consideration is to find new EAQMDS codes from other classical linear codes such as constacyclic codes and the generalized Reed-Solomon codes.

Acknowledgements The authors would like to thank the referees and editor in chief for their helpful comments and a very meticulous reading of this manuscript.

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